MMP Learning Seminar

Week 40:

Applications of the existence of minimal models for varieties of general type.

→ Flops connect minimal models.

→ Fano varieties are Mori dream spaces.

→ Canonical models of $M_{g,n}$

→ Applications to singularity theory.

Theorem: (X,Δ) projective klt. If Δ is big, then there exists a minimal model program with scaling of an ample divisor which terminates.

Corollary: (X,Δ) projective klt, Δ big & $K_X+\Delta$ is pseudo-effective. Then (X,Δ) admits a good minimal model

Theorem: (X,Δ) projective klt. Then

$$\bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$$

is finitely generated over $\mathbb{K}$.

(X,Δ) klt, Δ big.

$K_X+\Delta$ pseff. $\quad (X,\Delta) \overset{\text{MMP with scaling}}{\dashrightarrow} (X',\Delta')$ klt, Δ' big.

$K_{X'}+\Delta'$ is semiample.

$K_X+\Delta$ not pseff.

$$\begin{tikzcd} (X,\Delta) \arrow[r, dashed, "\text{MMP with scaling}"'] & X' \arrow[d] \\ & Z \end{tikzcd}$$

Mori fiber space

Flops connect minimal models:

$\pi : X \longrightarrow U$ proj morphism between g.p normal

(X, Δ) klt, Δ big over U.

$\phi_i : X \dashrightarrow Y_i$ for $i = 1, 2$, be two minimal models for (X, Δ) over U. Let $\Gamma_i = \phi_{i*} \Delta$.

X, Y_1, Y_2 assumed to be $\mathbb{Q}$-factorial.

Then $Y_1 \dashrightarrow Y_2$ is a composition of $(K_{Y_1} + \Gamma_1)$-flops over U.

Proof:

$$\begin{tikzcd} Y_1 \arrow[rr, dashed] \arrow[dr] \arrow[ddr] & & Y_2 \arrow[dl] \arrow[ddl] \\ & Z \arrow[d] & \\ & U & \end{tikzcd}$$

ample models

$K_{Y_1} + \Gamma_1$ is nef over U.

Γ_1 is big, (Y_1, Γ_1) klt

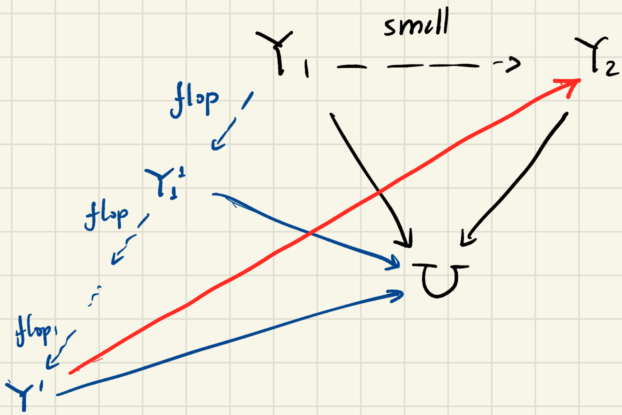
$K_{Y_1} + \Gamma_1 \equiv_Z 0$. & $K_{Y_2} + \Gamma_2 \equiv_Z 0$.

We may replace U with Z and assume

H_2 ample on Y_2 & H_1 its strict transform on Y_1.

$(Y_1, \Gamma_1 + H_1)$ is klt. assume $K_{Y_1} + \Gamma_1 + H_1$ is not nef.

We perform a $(K_{Y_1} + \Gamma_1 + H_1)$-flip over U which is a $(K_{Y_1} + \Gamma_1)$-flop.



$$Y' \xrightarrow{s} Y_2$$

small morphism between var.

$$K_{Y_1}+\Gamma_1 \equiv K_{Y_2}+\Gamma_2 \equiv_U 0.$$

$K_{Y_1'}+\Gamma_1'+H_1'$ is nef ?

yes $\longrightarrow$ stop.

no $\longrightarrow$ new flop.

$K_{Y'}+\Gamma_{Y'}+H_{Y'}$ is semiample

$K_{Y_2}+\Gamma_2+H_2$ is ample

is the ample model.

$\square$

Proof by Kawamata:

X & X' $\mathbb{Q}$-fact.

minimal models

$$
\begin{tikzcd}
 & & (X,B) \arrow[rr, dashed, "\text{small}"] \arrow[dl, dashed, "\text{flops}"'] \arrow[dr, "f"] & & (X',B') \arrow[dl, "f'"] \\
 & (X_1,B_1) \arrow[dl, dashed, "\text{flops}"'] & & S & \\
 (X_n,B_n) \arrow[uurrrr, "\sim"] & & & &
\end{tikzcd}
$$

L' ample on X'.

$(X, B+2L)$ klt.

(X_n, B_n+2L_n) klt &

$K_{X_n}+B_n+2L_n$ nef.

$k(K_X+B)$ is Cartier, $e = \dfrac{1}{2k\dim X+1}$.

$K_X+B+e2L$ is not nef $\Longrightarrow$

$(K_X+B+e2L)$-neg extremal ray which is also a (K_X+B+2L)-negative.

$$0 > ((K_X+B+2L)\cdot C) \geq -2\dim X.$$

Claim: $(K_X+B).C=0.$

Proof: Assume otherwise that $(K_X+B).C>0.$

Then $(K_X+B)\geq \frac{1}{k}.$

$$(K_X+B+e2L)\cdot C =$$

$$\frac{1}{2k\dim X+1}\overbrace{((K_X+B+2L).C)}^{\geq -2\dim X}+\frac{2k\dim X}{2k\dim X+1}\overbrace{((K_X+B).C)}^{\geq \frac{1}{k}.}$$

$$\geq \frac{1}{2k\dim X+1}(-2\dim X+2\dim X)=0$$

Remarks: The sequence of $\overset{K_X+B}{\text{flops}}$ that Kawamata constructs are obtained by a MMP with scaling of an ample divisor for $\underbrace{(X,\underbrace{B+e2L}_{\text{big}})}_{\text{klt}}.$

Fano varieties are Mori dream spaces:

Corollary: $\pi : X \longrightarrow U$ proj morphism.

$A \geq 0$ ample $\mathbb{Q}$-divisor over U. $\Delta_i = A + B_i$.

where $B_i \geq 0$ $\mathbb{Q}$-divisors. Assume (X, Δ_i) are dlt.

$K_X + \Delta_i = D_i$. Then the ring

$$R(\pi, D^\bullet) = \bigoplus_{m \in \mathbb{N}^k} \pi_* \mathcal{O}_X(\lfloor \textstyle\sum_i m_i D_i \rfloor)$$

is a finitely generated $\mathcal{O}_U$-module

Proof: $f : Y \longrightarrow X$ log resolution of all the (X, Δ_i)

$$K_Y + \underbrace{\Gamma_i}_{\geq 0} = \pi^*(K_X + \Delta) + \underbrace{E_i}_{\geq 0}$$

Assume is log smooth

A ample on X. F exceptional s.t. $f^*A - F$ ample on Y.

and $(Y, \Gamma_i + F)$ is klt. $A' \sim_{\mathbb{Q}} f^*A - F$ general ample

$$G_i = K_Y + \Gamma_i + F - f^*A + A' \sim_{\mathbb{Q}, U} K_Y + \Gamma_i.$$

$$R(\pi, D^\bullet) \text{ f.g} \Longleftrightarrow R(\pi \circ f, G^\bullet) \text{ f.g.}$$

These rings have isom truncation.

Replace X and Δ_i's with $\boxed{Y \text{ and } \Gamma_i\text{'s}}$
D_i's $\boxed{G_i\text{'s.}}$

$m\Delta_i$ Weil divisors. $E = \bigoplus_{i=1}^{k} \mathcal{O}_X(m\Delta_i)$

$Y = \mathbb{P}_X(E)$, $f : Y \longrightarrow X$.

$\sigma_i \in \mathcal{O}_X(m\Delta_i)$, with zero locus $m\Delta_i$, $\sigma = (\sigma_1, \ldots, \sigma_k) \in H^0(X, E)$

S the divisor of σ in Y. $T_1, \ldots, T_k$ sections of E.

$T = T_1 + \ldots + T_k$, $\Gamma = T + S/m$

$\mathcal{O}_Y(m(K_Y + \Gamma))$ is the tautological line bundle associated to
$E(mK_X)$.

Thus, $R(\pi, D^\bullet) \simeq R(\pi \circ f, m(K_Y + \Gamma))$.

$\hookrightarrow$ reduce to the case $k=1$.

Claim: $\begin{cases} \text{We need to check } \Gamma = \text{ample} + \text{eff} \\ \& \ (Y, \Gamma) \text{ is dlt.} \end{cases}$

(Y, Γ) is log smooth outside $\operatorname{supp} \Gamma$.

Adjuction + induction proves that (Y, Γ) is dlt around Γ.

$f^*A \leq S/m \leq \Gamma$. T ample over X.

Hence, $f^*A + \varepsilon T$ is ample on Y (over U).

$$A' \sim_{\mathbb{Q},U} f^*A + \varepsilon T \quad \text{general ample}$$

Then, we write:

$$\underset{\substack{\| \\ \text{ample + eff}}}{K_Y + \Gamma'} = K_Y + \underbrace{\Gamma - \varepsilon T}_{} - f^*A + A' \sim_{\mathbb{Q},U} K_Y + \Gamma$$

(Y, Γ') klt.

$$R(Y/U, m(K_Y + \Gamma)) \simeq R(Y/U, m(K_Y + \Gamma'))$$

$\square$

Corollary: $\pi: X \longrightarrow U$ projective, U affine.

X $\mathbb{Q}$-factorial, (X,Δ) dlt, $-(K_X+\Delta)$ ample over U.

Then X is a MDS.

Proof: $h^1(\mathcal{O}_X)=0$ (from KV).

$D_1,\ldots,D_k$ divisors generating $N^1(X)$.

$\Gamma \in |-m(K_X+\Delta)|$ general.

$(X,\Delta+\Gamma/m)$ klt $\quad K_X+\Delta+\Gamma/m \sim_{\mathbb{Q},U} 0$.

$(X,\Delta+\Gamma/m+\frac{1}{n}D_i)$ klt.

$n\left(K_X+\Delta+\Gamma/m+\frac{1}{n}D_i\right) \sim_{\mathbb{Q}} D_i$.

$\square$.

Moduli spaces of curves:

Corollary (1.2.1): Let $X = \overline{M}_{g,n}$.

Δ_i with $1 \leq i \leq k$ denote the boundary divisors.

$\Delta = \sum_i a_i \Delta_i$, $0 \leq a_i \leq 1$. Then (X, Δ) is log canonical. If $K_X + \Delta$ is big, then it has an ample model. If $a_i \geq \delta$, for some fixed δ, then the ample models obtained are only finitely many.

Lemma: $X = \overline{M}_{g,n}$, X is $\mathbb{Q}$-factorial and klt.

$D =$ reduced boundary. (X, D) log canonical and $K_X + D$ is ample.

Lemma: $X = \overline{M}_{g,n}$, X is $\mathbb{Q}$-factorial and klt.

D = reduced boundary. (X,D) log canonical and $K_X + D$ is ample.

Proof: (X,D) is locally the quotient of a normal crossing pair.

If $n=0$, then K_X+D is ample (Mumford, 1977)

$$\begin{tikzcd} \pi : \overline{M}_{g,n+1} \arrow[r] & \overline{M}_{g,n} \\ (Y,G) & (X,D) \end{tikzcd}$$

$$\begin{tikzcd} \Gamma \subseteq \overline{M}_{g,m+1} \arrow[r] \arrow[d, "\psi"'] & \overline{M}_{g,n+1} \arrow[d, "\pi"] \\ \overline{M}_{g,n} \arrow[r, "f"] & \overline{M}_{g,n} \end{tikzcd}$$

the universal morphism

definition of stable pairs, we get $K_{\overline{M}_{g,n+1}} + \Gamma$ has positive degree on the fibers of ψ.

Hence $K_Y + G$ is π-ample.

Towards ample cone of $\overline{M}_{g,n}$

Gibney, Keel, Morrison

We can write.

$$K_Y + G = \pi^*(K_X + D) + \psi \quad \text{where } \psi \text{ is nef}$$

$K_X + D$ is ample by induction on n.

$\varepsilon > 0$ small enough.

$$\varepsilon(K_Y + G) + (1-\varepsilon)\pi^*(K_X + D) \quad \text{ample}$$

Then

$$K_Y + G = \varepsilon(K_Y + G) + (1-\varepsilon)(K_Y + G)$$

$$= \underbrace{\underbrace{\varepsilon(K_Y + G) + (1-\varepsilon)\pi^*(K_X + D)}_{\text{ample}} + \underbrace{(1-\varepsilon)\psi}_{\text{nef}}}_{\text{ample}}.$$

$\square$

Proof of (1.2.1): $K_X + D$ is ample & log canonical.

Hence, $K_X + \Delta$ is klt provided $a_i < 1$.

Pick $\underset{\substack{\vee\!\!| \\ 0}}{A} \sim_{\mathbb{Q}} \delta(K_X + D)$ $\quad \boxed{a_i \geq \delta}$

general ample. Note that.

$$(1+\delta)(K_X + \Delta) = K_X + \underbrace{\delta(K_X+D)}_{} + \underbrace{(1+\delta)\Delta - \delta D}_{}$$

$$\sim_{\mathbb{Q}} K_X + A + \underset{\substack{\vee\!\!| \\ 0}}{B}$$

klt with boundary of the form $A + B_{\geq 0}$.

$$0 \leq (\Delta - \delta D) + \delta\Delta = B = \Delta + \delta(\Delta - D) \leq D.$$

$B \leq D.$

Then, we can apply finiteness of ample models. $\square$

Singularity Theory:

$(X; x)$ an algebraic sing.

$\varphi : Y \longrightarrow X$ projective birational

$\cup\!|$
E $\qquad$ φ is an isomorphism between $Y \setminus E \simeq X \setminus \{x\}$

Tackle question on $(X; x)$ by studying the projective variety E.
This is called a global-to-local principle

Corollary 1.4.3: Let (X, Δ) be a klt pair.

$\mathcal{C}$ be a finite set of divisorial valuations over X
with log discrepancies in the interval $(0, 1)$.
Then, we may find a projective birational morphism
$\pi : Y \longrightarrow X$, s.t Y is $\mathbb{Q}$-factorial and
the exceptional divisors of π correspond to elements of $\mathcal{C}$

Proof: $W \xrightarrow{f} X$ log resolution

extracting all the divisors corresponding to elements of $\mathcal{C}$

$$K_W + \overset{\geq 0}{\Psi} = f^*(K_X + \Delta) + E^{\geq 0}.$$

$$f_*\Psi = \Delta. \qquad \Psi \wedge E = 0.$$

F = sum of all prime divisors, exceptionals over X, neither on E nor $\mathcal{C}$.

$\Phi = \Psi + \varepsilon F$ for ε small enough

$$\boxed{K_W + \Phi} = K_W + \Psi + \varepsilon F.$$

has a good minimal model over X.

klt, Φ big over X.

$$\begin{tikzcd} W \arrow[r, dashed, "g"] \arrow[d] & Y \arrow[dl, "\pi"] \\ X & \end{tikzcd}$$

$$g_*\Phi = \Gamma, \qquad g_*(E + \varepsilon F) = \overset{\substack{0 \\ \parallel}}{E'}$$

$$\underbrace{K_Y + \Gamma}_{\text{nef over } X} = \underbrace{\pi^*(K_X + \Delta)}_{\text{trivial over } X} \cancel{+ E'}$$

The only divisors that we extract on Y are those in $\mathcal{C}$ $\square$